WEINER-HOPF EQUATION AND FREDHOLM PROPERTY OF THE GOURSAT PROBLEM IN GEVREY SPACES(Microlocal Geometry)

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WIENER–HOPF EQUATION AND FREDHOLM PROPERTY OF THE GOURSAT PROBLEM IN GEVREY SPACES

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0. Introduction

In this note, we will explain a part of a preprint [MY], where a Fredholm property of the Goursat problem in (formal) Gevrey space is characterized by the Hilbert factorizability of Toeplitz symbol associated with the Gevrey space in which the Goursat problem should be solved. To do so, the norm inequality for the (finite section) Toeplitz operator which holds under the Hilbert factorizability condition plays a cruicial role.

In the preceding papers ([M2], [MH]), Miyake studied the Goursat problem in Gevrey space, and made clear the meaning of Newton polygon associated with the operator which will be defined below. In these papers, he proved that the Gevrey index of Gevrey space in which the Goursat problem should be solved is determined from the slopes of sides of Newton polygon for various type of linear partial differential operators. The main purpose of these papers was to give a relation between the slopes of sides of Newton polygon and the Gevrey indices, and the analysis was done mainly for the vertices of Newton polygon and the solvability of the Goursat problem in Gevrey space of Roumieu type or of Beurling type was proved. Moreover, it was proved that if the Goursat problem is formulated from an interior point of a side of Newton polygon, then the Goursat problem is uniquely solvable under the so called spectral condition. On the other hand, if we remove the spectral condition, the problem turns to be more complicated, and we have to make a delicate anaysis of interior points of spectral radius. Such an analysis was first done by Leray for the Goursat problem in the category of local holomorphic functions by a simple example of equations ([L]) (See Example 1 in Section 2.) After the work of Leray, Yoshino published a series of papers on the spectral problem of the Goursat problem in the category of local holomorphic functions ([Ys1],

[Ys2], [Ys3], [Ys4], [Ys5]). Recently, Miyake has extended Leray's results to the Goursat problem in (formal) Gevrey space, and the spectral problem of the Goursat problem was interpreted clearly as a spectral problem of integro-differential operators on Banach space of (formal) Gevrey functions ([M3]).

The preprint [MY] intends to give a unification of these individual results by the Hilbert factorizablity condition of Toeplitz symbol for holomorphic linear partial differential operators defined in a neighbourhood of the origin of $\mathbf{C}^2$.

We remark that the Newton polygon defined below is an extension of Ramis' one ([R1], [R2]). In these papers, he studied the irregurality or the index theorem of ordinary differential operators on Gevrey space, and the Newton polygon plays a crucial role to describe and understand his results. It is the purpose of a series of papers ([M2], [M3], [MH] and [MY]) to give some analogues of Ramis' results to linear partial differential operators.

1. Statement of result

Let $P \equiv P(t, x; D_t, D_x)$ be a linear partial differential operator of finite order with holomorphic coefficients in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x$, and write it in the form,

$$P = \sum_{\sigma \in \mathbf{N}} \sum_{j,\alpha \in \mathbf{N}}^{\text{finite}} a_{\sigma j \alpha}(x)\, t^\sigma D_t^j D_x^\alpha, \tag{1.1}$$

where $\mathbf{N}$ denotes the set of nonnegative integers.

For a triplet $(\sigma, j, \alpha) \in \mathbf{N}^3$, we associate a left half line $Q(\sigma, j, \alpha)$ in a (u, v)-plane defined by,

$$Q(\sigma, j, \alpha) := \{(u, \sigma - j) \in \mathbf{R}^2\,;\, u \leq \alpha + j\}.$$

Then the Newton polygon $\mathrm{N}(P)$ of the operator P is defined by

$$\mathrm{N}(P) := \mathrm{ch}\,\{Q(\sigma, j, \alpha)\,;\, a_{\sigma j \alpha}(x) \not\equiv 0\}, \tag{1.2}$$

where $\mathrm{ch}\,\{\cdot\}$ denotes the convex hull.

We denote by $\mathbf{C}[[t, x]]$ the set of formal power series of variables $t, x \in \mathbf{C}$, and $\mathcal{O}(\Omega)$ the set of holomorphic functions on a complex domain Ω .

Let $s, w, R > 0$. Then we define $\mathcal{G}^s_w(R)$, a Gevrey space with an index s, by the following isomorphism of Frechét spaces,

$$\mathbf{C}[[t, x]] \supset \mathcal{G}^s_w(R) \xrightarrow[\sim]{\text{Borel transf.}} \mathcal{O}((|t|\,/w)^{1/s} + |x| < R), \tag{1.3}$$

where the Borel transformation is defined by

$$\mathcal{G}^s_w(R) \ni \sum_{j,\alpha\in\mathbf{N}} u_{j\alpha}\frac{t^j x^\alpha}{j!\alpha!} \mapsto \sum_{j,\alpha\in\mathbf{N}} u_{j\alpha}\frac{t^j x^\alpha}{(sj)!\alpha!} \in \mathcal{O}((|t|/w)^{1/s} + |x| < R).$$

The factorial is defined by the gamma function, $r! := \Gamma(r+1)$ for $r \geq 0$.

Remark 1 ([MY, Proposition 5.1]).

$u(t,x) = \sum u_{j\alpha} t^j x^\alpha / j!\alpha! \in \mathcal{G}^s_w(R) \Longleftrightarrow 0 < {}^\forall r < R,\ {}^\exists C(u,r) \geq 0$ s.t.

$$|u_{j\alpha}| \leq C(u,r)\,\frac{(sj+\alpha)!}{w^j\, r^{sj+\alpha}}. \tag{1.4}$$

This implies that $u_j(x) = \sum_{\alpha\in\mathbf{N}} u_{j\alpha} x^\alpha/\alpha! \in \mathcal{O}(|x| < R)$ (${}^\forall j \in \mathbf{N}$) and

$$u_j(x) \ll C(u,r)\,\frac{r}{w^j}\,\frac{(sj)!}{(r-x)^{sj+1}} \quad (0 < {}^\forall r < R),$$

where $\sum a_\alpha x^\alpha \ll \sum A_\alpha x^\alpha$ means that $|a_\alpha| \leq A_\alpha$ for ${}^\forall\alpha$. This is the reason why we call $\mathcal{G}^s_w(R)$ the Gevrey space with an index s. We define

$$\mathcal{G}^s_w := \bigcup_{R>0} \mathcal{G}^s_w(R)\ (0 < w < \infty), \quad \mathcal{G}^s_0 := \bigcup_{w>0} \mathcal{G}^s_w, \quad \mathcal{G}^s_\infty := \bigcap_{w>0} \mathcal{G}^s_w. \tag{1.5}$$

It is often that $\mathcal{G}^s_0$ (resp. $\mathcal{G}^s_\infty$) is called of Roumieu type (resp. of Beurling type).

We consider the following Goursat problem in $\mathcal{G}^s_w$ with zero Goursat data,

$$\begin{cases} P\,u(t,x) = f(t,x) \in \mathcal{G}^s_w, \\ u(t,x) = O(t^l x^\beta) \quad \text{in} \quad \mathcal{G}^s_w, \end{cases} \tag{G}$$

where $u(t,x) = O(t^l x^\beta)$ in $\mathcal{G}^s_w$ means that $t^{-l}x^{-\beta}u(t,x) \in \mathcal{G}^s_w$. It is the same to consider the following mapping,

$$P : t^l x^\beta \mathcal{G}^s_w \longrightarrow \mathcal{G}^s_w. \tag{G}$$

In order to study the problem (G), let us define the principal part and the Toeplitz symbol as follows.

For a given $s > 0$, we draw a line L_s in the plane with slope $k := 1/(s-1) \in \mathbf{R}\cup\{\infty\}$ which contacts on a side or at a vertex of $\mathrm{N}(P)$. Therefore, when $0 < s < 1$ it is assumed, a priori, that the operator P has polynomial coefficients in the variable t.

We put $N_s := \mathrm{N}(P) \cap L_s$, and

$$\overset{\circ}{N}_s := \{(j,\alpha) \in \mathbf{N}^2\,;\, a_{0j\alpha}(0) \neq 0,\ (j+\alpha,-j) \in N_s\}.$$

As a fundamental hypothesis, we assume the following condition thrughout this note.

$$\text{(A)} \qquad \overset{\circ}{N}_s \neq \phi.$$

The principal part $P_s \equiv P_s(D_t, D_x)$ and the Toeplitz symbol $f_s(z)$ of the problem (G) are defined by,

$$(1.6) \qquad P_s := \sum_{(j,\alpha)\in\overset{\circ}{N}_s} a_{0j\alpha}(0)\, D_t^j D_x^\alpha, \quad f_s(z) := \sum_{(j,\alpha)\in\overset{\circ}{N}_s} a_{0j\alpha}(0)\, z^{-j}.$$

Now our result in this note is the following,

Theorem A. *Let* $(l,\beta) \in \mathbf{N}^2$ *belong to* $\mathrm{ch}\,\{\overset{\circ}{N}_s\}$, *and assume*

$$(\mathrm{H})_w \qquad f_s(z) \neq 0 \text{ on } |z| = w, \text{ and } \oint_{|z|=w} d(\log f_s(z) z^l) = 0.$$

Then the Goursat problem (G) *has a Fredholm property, that is, the mapping* $P : t^l x^\beta \mathcal{G}_w^s \longrightarrow \mathcal{G}_w^s$ *has the same finite dimensional kernel and cokernel. Furthermore, if one of the following conditions is satisfied, then the problem* (G) *is uniquely solvable in* $\mathcal{G}_w^s$:

(i) (l,β) *is an end point of* $\mathrm{ch}\,\{\overset{\circ}{N}_s\}$.

(ii) *There exists* $c > 0$ *such that* $\{f_s(z)z^l\,;\, |z| = c\}$ *is a segment.*

(iii) *There exists* $c > 0$ *such that* $0 \notin \mathrm{ch}\,\{f_s(z)z^l\,;\, |z| = c\}$.

Moreover, every formal solution $u(t,x) \in \mathbf{C}[[t,x]]$ *of the problem* (G) (*if it exists*) *belongs to* $\mathcal{G}_w^s$. *Precisely, the mapping*

$$P : t^l x^\beta \mathbf{C}[[t,x]]/t^l x^\beta \mathcal{G}_w^s \longrightarrow \mathbf{C}[[t,x]]/\mathcal{G}_w^s$$

is bijective.

Remark 2. (i) If s is an irrational, then $\overset{\circ}{N}_s$ consists of a point whenever the assumption (A) is satisfied, and the problem (G) is uniquely solvable in $\mathcal{G}_w^s$ for every $0 \leq w \leq \infty$. Since this case was studied precisely in [MH], our interest is in the case where $s \in \mathbf{Q}$ and $\overset{\circ}{N}_s$ includes at least two elements.

(ii) When (l,β) is a lower (resp. an upper) end point of $\mathrm{ch}\,\{\overset{\circ}{N}_s\}$, the problem (G) is uniquely solvable in $\mathcal{G}_0^s$ (resp. in $\mathcal{G}_\infty^s$). Because in this case $f_s(z)z^l \in \mathbf{C}[z]$ (resp. $\in \mathbf{C}[z^{-1}]$) with non zero constant term and the condition $(\mathrm{H})_w$ is satisfied for sufficiently small $w > 0$ (resp. large $w > 0$). In the case where (l,β) is an interior point of $\mathrm{ch}\,\{\overset{\circ}{N}_s\}$, it is known that the problem (G) is uniquely solvable in $\mathcal{G}_w^s$ under the so called spectral condition ([MH, Theorem B], see also [W], [M1] and [M2]));

$$\text{(S)} \qquad |a_{0l\beta}(0)| > \sum_{(j,\alpha)\in \overset{\circ}{N}_s \setminus (l,\beta)} |a_{0j\alpha}(0)|\, w^{l-j}.$$

(iii) If $s \geq 1$, we may assume that the coefficients of the operator belong to $\mathcal{G}_w^s$.

The proof of Theorem A is long, so we give only a course of the proof in the below. (See [MY] for detail. In this preprint, the case of nonpositive Gevrey index is also studied.)

2. Reduction to the spectral problem of an integro-differential operator

Composing the following mappings,

$$\mathcal{G}_w^s(R) \xrightarrow[\sim]{D_t^{-l}D_x^{-\beta}} t^l x^\beta \mathcal{G}_w^s(R) \xrightarrow{P} \mathcal{G}_w^s(R),$$

Theorem A is proved by converting the Goursat problem or the mapping (G) to the Fredholm property of an integro-differential operator $L := P\,D_t^{-l}D_x^{-\beta}$ on a Banach space $G_w^s(R)$ associated with the space $\mathcal{G}_w^s(R)$ which is defined as follows.

We put $U(t,x) = \sum U_{j\alpha} t^j x^\alpha / j!\,\alpha! \in \mathbf{C}[[t,x]]$. Then we define

$$\text{(2.1)} \qquad U(t,x) \in G_w^s(R) \overset{\text{def}}{\iff} \|U\|_{w,R}^{(s)} := \sum_{j,\alpha\in\mathbf{N}} |U_{j\alpha}| \frac{w^j R^{sj+\alpha}}{(sj+\alpha)!} < \infty.$$

It should be mentioned that $G_w^s(R)$ is a Banach algebra when $s \geq 1$ which implies the fact (iii) in Remark 2. From (1.4) we see that it holds that

$$\text{(2.2)} \qquad \mathcal{G}_w^s(R) = \bigcap_{0<r<R} G_w^s(r)\,.$$

The above consideration shows that it is sufficient to study the Fredholm property of the following mapping,

$$\text{(2.3)} \qquad L : G_w^s(R) \longrightarrow G_w^s(R).$$

For this purpose, we rewrite the operator $L = P\, D_t^{-l} D_x^{-\beta}$ as

(2.4)
$$L = \sum_{\sigma\in\mathbf{N}} \sum_{j,\alpha\in\mathbf{Z}}^{\text{finite}} a_{\sigma j\alpha}(x)\, t^{\sigma} D_t^j D_x^{\alpha},$$

where $\mathbf{Z}$ denotes the set of integers. The symbol D_t^{-1} denotes the integration in the variable t from 0 to t in the formal sence, and it is the same for D_x^{-1}.

By the condition for the Newton polygon of the operator P, we know that the Newton polygon $\mathrm{N}(L)$ of the operator L has a side N_s with slope $k = 1/(s-1)$ which includes the origin. This assumption implies that

(2.5)
$$\begin{cases} sj + (1-s)\sigma + \alpha \le 0 \quad \text{if} \quad a_{\sigma j\alpha}(x) \not\equiv 0, \\ sj + (1-s)\sigma + \alpha = 0 \; (a_{\sigma j\alpha}(x) \not\equiv 0) \quad \text{if and only if} \quad (j+\alpha, \sigma - j) \in N_s. \end{cases}$$

We note that the assumption (A) is equivalent to

$$\overset{\circ}{N}_s \equiv \{(j,\alpha) \in \mathbf{Z}^2\,;\, a_{0j\alpha}(0) \ne 0,\; (j+\alpha, -j) \in N_s\} \ne \phi,$$

and $\mathrm{ch}\,\{\overset{\circ}{N}_s\} \ni 0$. The principal part $L_0 \equiv L_0(D_t, D_x)$ and the Toeplitz symbol $f(z)$ of the mapping (2.3) are defined by

$$L_0 = P_s\, D_t^{-l} D_x^{-\beta}; \quad f(z) = f_s(z) z^l.$$

The condition $(\mathrm{H})_w$ is rewritten in the form,

$(\mathrm{H})_w$
$$f(z) \ne 0 \text{ on } |z| = w, \text{ and } \oint_{|z|=w} d(\log f(z)) = 0.$$

Remark 3. The condition $(\mathrm{H})_w$ is equivalent that the Toeplitz symbol $f(z)$ is decomposed into $f(z) = f_+(z)\, f_-(z)$, where $f_+(z)$ (resp. $f_-(z)$) is holomorphic and does not vanish on $\{z \in \mathbf{C}\,;\, |z| \le w\}$ (resp. on $\{z \in \mathbf{C}\,;\, w \le |z| \le \infty\}$). Such a decomposition is called a *Hilbert factorization* of the symbol $f(z)$ with respect to a circle $\{z \in \mathbf{C}\,;\, |z| = w\}$.

Let us decompose the operator L as follows.

(2.6)
$$L(t,x;D_t,D_x) = L_0(D_t,D_x) + L_1(t,x;D_t,D_x) + L_2(t,x;D_t,D_x),$$

where

$$L_1 = \sum_{sj+(1-s)\sigma+\alpha=0} a_{\sigma j\alpha}(x) t^{\sigma} D_t^j D_x^{\alpha} \quad (\sigma > 0 \text{ or } a_{\sigma j\alpha}(0) = 0),$$

$$L_2 = \sum_{sj+(1-s)\sigma+\alpha<0}^{\text{finite}} a_{j\sigma\alpha}(t,x) t^{\sigma} D_t^j D_x^{\alpha}.$$

The following facts are fundamental ([MY, Lemmas 4.2, 4.3]): There exists a positive constant R_0 such that

(i) L_2 is a compact operator on $G^s_w(R)$ $(w > 0, 0 < R < R_0)$.

(ii) L_j $(j = 1, 2)$ are bounded operators on $G^s_w(R)$ $(0 < R < R_0)$ and their operator norms are estimated by $\|L_j\| = o(1)$ as $R \to 0$.

Now Theorem A follows from the following,

Theorem B. *Under the assumption* (A), L_0 *is a Fredholm operator on* $G^s_w(R)$ *if and only if the condition* $(\mathrm{H})_w$ *is satisfied, and the index is always equal to* 0. *Moreover, if one of the following conditions is satisfied, then* L_0 *is invertible* :

(i) $f(z)$ *is a polynomial of* z *or* z^{-1}. *This is the case where the origin is an end point of* $\mathrm{ch}\,\{\overset{\circ}{N}_s\}$.

(ii) *There exists* $c > 0$ *such that* $\{f(z)\,;\,|z| = c\}$ *is a segment. In this case the set of eigenvalues of* L_0 *is dense on this segment, and the resolvent set of the operator* L_0 *on* $G^s_w(R)$ *is independent of* $R > 0$ *(which we denote by* $\rho(L_0\,;\,G^s_w)$*) and is given by,*

$$\rho(L_0\,;\,G^s_w) = \{\lambda \in \mathbf{C}\,;\,(\mathrm{H})_w \text{ is satisfied for } f(z) - \lambda\} =: \Gamma_f(w). \tag{2.7}$$

(iii) *There exists* $c > 0$ *such that* $0 \notin \mathrm{ch}\,\{f(z)\,;\,|z| = c\}$. *In this case the set of eigenvalues of* L_0 *which is contained in* $\Gamma_f(w)$ *consists of finite points.*

As a corollary to this Theorem and the fundamental facts cited above, we have the following,

Corollary. *Suppose* $L_1 = 0$ *in the decomposition of* L. *Then* L *is a Fredholm operator on* $G^s_w(R)$ *for some* $R > 0$ *if and only if the condition* $(\mathrm{H})_w$ *is satisfied. Furthermore we have*

$$\rho(L_0\,;\,G^s_w) = \bigcup_{R>0} \rho(L\,;\,G^s_w(R)), \tag{2.8}$$

where $\rho(L\,;\,G^s_w(R))$ *denotes the resolvent set of the operator* L *on* $G^s_w(R)$.

Here we give some examples.

Example 1 ([L],[M2]). Let $L_0 = D_t^p D_x^{-p-\alpha} + D_t^{-p} D_x^{p+\alpha}$ $(p \geq 1, p + \alpha > 0)$. Then the Gevrey index of this operator is $s = 1 + (\alpha/p) > 0$, and the Toeplitz symbol is $f(z) = z^{-p} + z^p$. Since $\{f(z)\,;\,|z| = 1\} = [-2, 2]$, by (ii) in Theorem B we have

$$\begin{aligned}\rho(L_0\,;\,G^s_w) &= \mathbf{C}\backslash\mathrm{ch}\{f(z)\,;\,|z| = w\} \\ &= \left\{\lambda \in \mathbf{C}\,;\,\left(\frac{\mathrm{Re}\lambda}{w^p + w^{-p}}\right)^2 + \left(\frac{\mathrm{Im}\lambda}{w^p - w^{-p}}\right)^2 > 1\right\} (\subset \mathbf{C}\backslash[-2, 2]).\end{aligned}$$

The set of eigenvalues $\sigma_p(L_0\,;\,G^s_w)$ is dense on $[-2,2]$, and is given exactly by

$$\sigma_p(L_0\,;\,G^s_w) = \bigcup_{n=0}^{\infty} \{2\cos\pi\theta\,;\,\sin(n+2)\pi\theta = 0,\ 0<\theta<1\}.$$

In [M2] the case $w=1$ was studied, and it was proved that $\rho(L_0\,;\,G^s_1) = \mathbf{C}\backslash[-2,2]$ $(=\bigcup_{w>0}\rho(L_0\,;\,G^s_w))$.

Next, let us consider the following equation in $\mathcal{G}^s_0$.

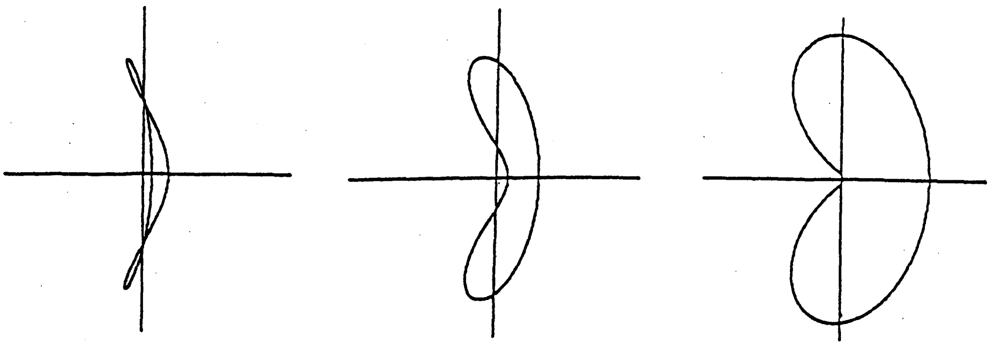
$$\{\lambda - L_0\}U(t,x) = F(t,x) \in \mathcal{G}^s_0. \tag{2.9}$$

It is easily seen that the equation is uniquely solvable in $\mathcal{G}^s_0$ if $\lambda \in \mathbf{C}\backslash[-2,2]$. In the case where $\lambda\in[-2,2]$, the situation is somewhat complicated. For $\lambda \in (-2,2)$, we set $\lambda = 2\cos\pi\theta$ $(0<\theta<1)$ and introduce an auxiliary function $\rho(\lambda)$ by

$$\rho(\lambda) := \liminf_{\mathbf{N}\ni n} |\sin(n\pi\theta)|^{1/n}. \tag{2.10}$$

Then the equation is uniquley solvable in $\mathcal{G}^s_0$ if and only if $\rho(\lambda)>0$ or $\lambda=\pm 2$. Leray and Pisot ([LP]) proved that the set of $\lambda\in(-2,2)$ such that $\rho(\lambda)=0$ is uncountable with Lebegue measure 0. Therefore, for an irratinal θ $(0<\theta<1)$ such that $\rho(\lambda)=0$, the Fredholm property does not hold for the equation (2.9). Indeed, the uniqueness of solutions does not imply the solvability. We note that such a phenomenon was first discovered by Leray [L] for the operator $L_0 = D_tD_x^{-1} + D_t^{-1}D_x$ on $\mathcal{G}^1_0$.

Example 2. Let $L_0 = \frac{4}{3}D_t^{-2}D_x^{2+2\alpha} + 3D_t^{-1}D_x^{1+\alpha} - \frac{9}{4}D_tD_x^{-1-\alpha}$, where $\alpha\ge 1$. Then the Gevrey index is given by $s = 1+\alpha \ge 2$ and the Toeplitz symbol is given by $f(z) = \frac{4}{3}z^2 + 3z - \frac{9}{4}z^{-1} = (\frac{4}{3}-z^{-1})(z+\frac{3}{2})^2$. This implies taht the condition $(\mathrm{H})_w$ is satisfied for $\frac{3}{4}<w<\frac{3}{2}$. Therefore, L_0 is a Fredholm operator on $G^s_w(R)$ for such w. In this case, it is easy to see that $0\in\bigcap_{3/4<w<3/2}\mathrm{ch}\,\{f(z)\,;\,|z|=w\}$, and L_0 has $1+\alpha$ dimensional kernel and cokernel on $G^s_w(R)$.



w=0.8　　w=1　　w=1.3

We return to the proof of Theorem B. Let us define an ideal $\mathcal{M}^s[N]$ of $\mathbf{C}[[t,x]]$ by

$$\mathcal{M}^s[N] := \{U(t,x)\in\mathbf{C}[[t,x]]\,;\,U_{j\alpha}=0 \text{ for } sj+\alpha<N\},$$

where $U(t,x)=\sum U_{j\alpha}t^j x^\alpha/j!\alpha!$, and define $G_w^s(R)[N]:=G_w^s(R)\cap\mathcal{M}^s[N]$. It is easy to see that L_0 (and also L) maps $\mathcal{M}^s[N]$ into itself.

Now we consider the following property:

(P) There exists N such that L_0 is invertible on $G_w^s(R)[N]$ and also on $\mathcal{M}^s[N]$.

Then Theorem B follows from the following,

Theorem C. *The property (P) holds if and only if the condition* $(\mathrm{H})_w$ *is satisfied. Therefore, if the condition* $(\mathrm{H})_w$ *is satisfied, then L_0 is bijective on $\mathbf{C}[[t,x]]/G_w^s(R)$.*

3. Wiener-Hopf equation and Theorem C

Theorem C is proved by the finite section Wiener-Hopf technique. We shall explain how the problem is reduced to the finite section Wiener-Hopf equation.

We define a Banach space $l_{1,w}$ by

$$l_{1,w}:=\{u(z)=\sum_{j=-\infty}^{\infty}u_j z^j\,;\,\|u\|_{1,w}:=\sum_{j=-\infty}^{\infty}|u_j|\,w^j<\infty\}. \tag{3.1}$$

We, sometimes, identify $u(z)\in l_{1,w}$ with a sequence $u=\{u_j\}_{j\in\mathbf{Z}}$ with the above defined norm. We denote by $l_{1,w}^+$ the set of $u(z)\in l_{1,w}$ with $u_j=0$ for $j<0$, and the projection $P:l_{1,w}\longrightarrow l_{1,w}^+$ is defined naturally.

Let $f(z)=\sum_{j=-m}^{n}f_j\,z^j\in\mathbf{C}[z,z^{-1}]$, the set of polynomials of z and z^{-1}. Then the Wiener-Hopf equation on $l_{1,w}^+$ with symbol $f(z)$ is defined by

$$P_f[u]:=P(fu)=g(z)\in l_{1,w}^+\quad(u(z)\in l_{1,w}^+). \tag{3.2}$$

The operator P_f is called a Toeplitz operator with symbol $f(z)$. The matrix represen-taion T_f of the operator P_f is given by

$$T_f:=(f_{j-k})_{j,k=0,1,2,\cdots}, \tag{3.3}$$

and T_f is called a Toeplitz matrix associated with $f(z)$. The Wiener-Hopf equation is rewritten in the form,

$$T_f\,u=g\in l_{1,w}^+\quad(u\in l_{1,w}^+). \tag{3.2$'$}$$

It is obvious that T_f defines a bounded operator on $l_{1,w}^+$ with operator norm $\|T_f\|=\|f\|_{1,w}$. Suppose that the symbol $f(z)$ is decomposed into $f(z)=f_+(z)f_-(z)$, where $f_+(z)\in\mathbf{C}[z]$ and $f_-(z)\in\mathbf{C}[z^{-1}]$. Then we have the following decomposition of T_f,

$$T_f=T_{f_+}T_{f_-}, \tag{3.4}$$

where T_{f_+} (resp. T_{f_-}) is a lower (resp. an upper) triangle matrix of infinite order. This decomposition shows that if $f_\pm(z)^{-1} \in l_{1,w}$, then T_f is invertible on $l^+_{1,w}$ and the inverse matrix is given by

$$T_f^{-1} = T_{f_-^{-1}} T_{f_+^{-1}} (= T_{f_-}^{-1} T_{f_+}^{-1}). \tag{3.5}$$

The operator norm of T_f^{-1} is estimated by

$$\|T_f^{-1}\| \le \|T_{f_-^{-1}}\| \|T_{f_+^{-1}}\| = \|f_-^{-1}\|_{1,w} \, \|f_+^{-1}\|_{1,w}. \tag{3.6}$$

Thus we have seen that if the Toeplitz symbol $f(z)$ satisfies the condition $(\mathrm{H})_w$ which means that $f(z)$ is Hibert factorizable with respect to the circle $\{z \in \mathbf{C}\,;\, |z| = w\}$, then T_f is invertible on $l^+_{1,w}$. The important fact is that the converse does hold.

Proposition 3.1. *The toeplitz matrix T_f is invertible on $l^+_{1,w}$ if and only if the condition $(\mathrm{H})_w$ is satisfied. Moreover, suppose that $f(z) \neq 0$ on $\{z \in \mathbf{C}\,;\, |z| = w\}$ and*

$$I_w(f) := \frac{1}{2\pi i} \oint_{|z|=w} d(\log f(z)) = k \neq 0. \tag{3.7}$$

If $k > 0$ then T_f is injective with k dimensional cokernel, and if $k < 0$ then T_f is surjective with $-k$ dimensional kernel.

For the proof, see [MY, Proposition 2.1]. We note that this result was proved by Calderón, Spitzer and Widom ([CSW]), in the case of l^∞ space.

Next, we consider the finite section Wiener-Hopf equation defined as follows. Let $u^{(N)}(z) = \sum_{j=0}^N u_j z^j$ and $g^{(N)}(z) = \sum_{j=0}^N g_j z^j$ for $N \in \mathbf{N}$. Then the N-th finite section Wienr=hopf equation with symbol $f(z)$ is defined by

$$P_f[u^{N)}] - g^{(N)}(z) = O(z^{N+1}), \tag{3.8}$$

where $O(z^{N+1})$ denotes the formal power series with power greater than N. We introduce an N-th finite section Toeplitz matrix, $T_f(N)$ by

$$T_f(N) := (f_{j-k})_{j,k=0,1,2,\cdots,N}. \tag{3.9}$$

Then the equation (3.8) is written by

$$T_f(N)\, u^{(N)} = g^{(N)} \in \mathbf{C}^{N+1}. \tag{3.8$'$}$$

For $u^{(N)} = {}^t(u_0, u_1, u_2, \cdots, u_N) \in \mathbf{C}^{N+1}$, we take an induced norm $\|u\|_{1,w;N}$ from $l^+_{1,w}$, that is,

$$\|u\|_{1,w;N} := \sum_{j=0}^N |u_j|\, w^j. \tag{3.10}$$

We denote by $l_{1,w}[N]$ the space $\mathbf{C}^{N+1}$ equipped with this norm.

Now the following proposition plays a key role in the proof of our result.

Proposition 3.2. *Let $f(z)$ satisfy the condition* $(\mathrm{H})_w$*. Then there exists $N_0 \in \mathbf{N}$ such that for any $N \geq N_0$ the N-th finite section Wiener-Hopf equation is uniquely solvable and the following norm inequality holds.*

$$\|u^{(N)}\|_{1,w;N} \leq K\|g^{(N)}\|_{1,w;N} \tag{3.11}$$

for a positive constant K independent of N. Conversely, this norm inequality holds only if the condition $(\mathrm{H})_w$ *is satisfied.*

We follow the argument of Baxter ([B]) for the proof of the norm inequality under the condition $(\mathrm{H})_w$. For the proof, see [MY, Proposition 2.3]. We remark that the condition $(\mathrm{H})_w)$ does not control every N. Concerning this, we can prove the following,

Proposition 3.3. *If $0 \notin ch\{f(z)\,;\,|z| = c\}$ for some $c > 0$, then $T_f(N)$ is ivertible for every $N \in \mathbf{N}$. If there exists $c > 0$ such that $\{f(z)\,;\,|z| = c\}$ is a segment, then the set of eigenvalues of $T_f(N)$ is dense in this segment.*

We note that the latter half is known as Szegö's Theorem ([GS]).

We are now in a position to explain how our problem is reduced to the results.

Let $L_0 = L_0(D_t, D_x)$ be the principal part, and $s \in \mathbf{Q}$ be the Gevrey index of L_0. Remember that we are interested in the case where s is a rational number and $\overset{\circ}{N}_s$ includes at least two elements. Let $s = q/p$ be an irreducible fraction of s. We notice that $(j,\alpha) \in \overset{\circ}{N}_s$ only if $sj + \alpha = 0$. Therefore, the principal part L_0 and the Toeplitz symbol $f(z)$ are rewritten by

$$L_0 = \sum_{j=-m}^{n} f_j D_t^{-jp} D_x^{jq}, \quad f(z) = \sum_{j=-m}^{n} f_j\, z^{jp}. \tag{3.12}$$

According to the irreducible fraction $s = q/p$, we decompose the Banach space $G_w^s(R)$ into an infinite direct product of finite section spaces as follows. We choose a lattice point $(l,\beta) \in \mathbf{N}^2$ such that $l - p < 0$, and put $d(l,\beta) = \max\{j\,;\,\beta - qj \in \mathbf{N}\}$. Let $U(t,x) = \sum U_{j\alpha} t^j x^\alpha / j!\alpha! \in G_w^s(R)$. We define a vector $\mathcal{U}^{(l,\beta)} \in l_{1,w}[d(l,\beta)]$ by

$$\mathcal{U}^{(l,\beta)} := {}^t(U_{l,\beta}, U_{l+p,\beta-q}, \cdots, U_{l+d(l,\beta)p,\beta-d(l,\beta)q}). \tag{3.13}$$

Then we have

$$\|U\|_{w,R}^{(s)} = \sum_{l,\beta}{}' \|\mathcal{U}^{(l,\beta)}\|_{1,w;d(l,\beta)} \frac{w^l R^{sl+\beta}}{(sl+\beta)!}, \tag{3.14}$$

where $\sum'_{l,\beta}$ denotes the summation over such l, β mentioned above. Thus the Banach space $G^s_w(R)$ is decomposed into an infinite direct product of finite section spaces.

According to this decomposition of the Banach space, the equation $L_0U(t,x) = F(t,x) \in G^s_w(R)$ is decomposed an ininite direct product of finite section Wiener-Hopf equations as follows. Let

$$T_{l,\beta} := (f_{j-k})_{j,k=0,1,2,\cdots,d(l,\beta)},$$

be the $d(l,\beta)$-th finite section Toeplitz matrix with symbol $f(z) = \sum_{j=m}^{n} f_j z^j$, where (l,β) is taken as above. Then the equation $L_0U(t,x) = F(t,x) \in G^s_w(R)$ is decomposed into finite section equations,

$$T_{l,\beta}\,\mathcal{U}^{(l,\beta)} = \mathcal{F}^{(l,\beta)} \in l_{1,w}[d(l,\beta)], \tag{3.15}$$

where $\mathcal{F}^{(l,\beta)}$ is defined from $F(t,x)$ similarly.

Therefore, we see that the property (P) holds if and only if there exists $N \in \mathbf{N}$ such that for any (l,β) such that $d(l,\beta) \geq N$ $T_{l,\beta}$ is invertible on $l_{1,w}[d(l,\beta)]$ and the norm inequality (3.11) holds for a positive constant K independent of l, β.

Summing up these results we have the following Theorem which is a precision of Theorem C.

Theorem D. *The following statements are equivalent* :

(i) *The property (P) holds.*

(ii) *The Toeplitz matrix T_f is invertible on $l_{1,w}$.*

(iii) *There exists N such that the norm inequality*

$$\|\mathcal{U}^{(l,\beta)}\|_{1,w;d(l,\beta)} \leq K\|\mathcal{F}^{(l,\beta)}\|_{1,w;d(l,\beta)} \tag{3.16}$$

holds for every l, β such that $d(l,\beta) \geq N$, where K is a positive constant independent of l, β.

(iv) *The Toeplitz symbol $f(z)$ is Hilbert factorizable with respect to the circle $\{z \in \mathbf{C}\,;\, |z| = w\}$.*

(v) *The condition* (H)$_w$ *is satisfied.*

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